Approximation problems for semigroups of operators and evolution equations

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joint work with F. Altomare and M. Cappelletti Montano (Univ. of Bari, Italy), and with I. Rașa (Tech. Univ. Cluj-Napoca, Romania)

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Statement of the problem

Let K be a convex compact subset of $\mathbb{R}^d$ ($d \geq 1$) having non-empty interior and consider elliptic second-order differential operators of the type

$$V(u) = \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \beta_i \frac{\partial u}{\partial x_i} + \gamma u \quad (u \in C^2(K)), \tag{1}$$

with coefficients $\alpha_{ij}, \beta_i, \gamma \in C(K)$.

As we shall see, under suitable hypotheses, the differential operator $(V, C^2(K))$ is closable and its closure $(A, D(A))$ generates a strongly continuous semigroup (briefly, C_0-semigroup) $(S(t))_{t\geq 0}$ on $C(K)$.

Thanks to the strong relationship between C_0-semigroups and linear evolution equations, this means that the abstract Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A(u(\cdot,t))(x) & x \in K\,, t \geq 0\,, \\ \\ u(x,0) = u_0(x) & u_0 \in D(A), x \in K \end{cases} \tag{2}$$

has a unique solution given by

$$u(x,t) = S(t)(u_0)(x) \quad (x \in K\,, t \geq 0), \tag{3}$$

and it continuously depends on the initial datum.

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We are interested on the problem of the constructive approximation of $(S(t))_{t\geq 0}$ in the following sense:

find a sequence $(B_n)_{n\geq 1}$ of bounded linear operators defined on $C(K)$ such that, if $t \geq 0$ and $(k_n)_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k_n/n = t$,

$$S(t)f = \lim_{n\to\infty} B_n^{k_n}(f) \quad \text{uniformly on } K, \tag{4}$$

for every $f \in C(K)$, $B_n^{k_n}$ being the iterate of B_n of order k_n.[1]

Formula (4), together with (3), yields

$$u(x,t) = S(t)u_0(x) = \lim_{n\to\infty} B_n^{k_n}(u_0)(x) \quad \text{uniformly with respect to } x \in K. \tag{5}$$

This problem, first stated in

F. Altomare, *Limit semigroups of Bernstein-Schnabl operators associated with positive projections*, Ann. Sc. Norm. Pisa, Cl. Sci., **16** (4) (1989), no. 2, pp. 259–279.

has been tackled in the last two decades from different angles and in different function spaces defined on a convex compact subset K of (a not necessarily finite dimensional) locally convex space as well as on noncompact real intervals.

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Differential operators associated with Markov operators

From now on let T be a Markov operator on $C(K)$, that is a positive linear operator from $C(K)$ into itself such that $T(\mathbf{1}) = \mathbf{1}$, $\mathbf{1}$ being the constant function of constant value 1 on K. Moreover, suppose that T satisfies the following condition

$$T(h) = h \quad \text{for each } h \in \{pr_1, \ldots, pr_d\}, \tag{6}$$

where, for any $i = 1, \ldots, d$, pr_i is the i-th coordinate function on K (i.e., $pr_i(x) := x_i$ for every $x = (x_1, \ldots, x_d) \in K$).

Given such a T, it is possible to construct an elliptic second-order differential operator W_T defined by setting, for every $u \in C^2(K)$,

$$W_T(u) = \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{7}$$

with, for each $i, j = 1, \ldots, d$, and $x \in K$,

$$\alpha_{ij}(x) = T(pr_i pr_j)(x) - pr_i pr_j(x) = T((pr_i - x_i)(pr_j - x_j))(x). \tag{8}$$

- W_T degenerates on the set

$$\partial_T K = \{x \in K \mid T(f)(x) = f(x) \text{ for every } f \in C(K)\} \tag{9}$$

of all interpolation points for T, which contains the set $\partial_e K$ of the extreme points of K.
- The boundary ∂K of K is generally non-smooth, due to the presence of possible sides and corners.

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Examples of W_T

1. Let $T : C([0,1]) \to C([0,1])$ be the positive linear operator

$$T(f)(x) = (1-x)f(0) + xf(1) \quad (f \in C([0,1]), 0 \le x \le 1). \tag{10}$$

Note that

$$T(h) = h \quad \text{for every } h \in \{\mathbf{1}, e_1\}$$

(here $e_1(t) = t$, $0 \le t \le 1$).

In this case, for every $u \in C^2([0,1])$ and $0 \le x \le 1$,

$$W_T(u)(x) = \frac{1}{2}x(1-x)u''(x)\,. \tag{11}$$

Observe that W_T degenerates at 0 and 1.

Examples of W_T

2. Given a function $b \in C([0,1])$ such that $0 \leq b(x) \leq \min\{2x, 2(1-x)\}$ for each $x \in [0,1]$, consider the positive linear operator $T : C([0,1]) \to C([0,1])$ defined by setting, for every $f \in C([0,1])$ and $0 \leq x \leq 1$,

$$T(f)(x) = \left(1 - x - \frac{b(x)}{2}\right) f(0) + b(x) f\left(\frac{1}{2}\right) + \left(x - \frac{b(x)}{2}\right) f(1). \tag{12}$$

T is such that $T(h) = h$ for each $h \in \{\mathbf{1}, e_1\}$ and, for every $C^2([0,1])$ and $0 \leq x \leq 1$,

$$W_T(u)(x) = \frac{1}{2}\left(x(1-x) - \frac{b(x)}{4}\right) u''(x)\,. \tag{13}$$

Note that W_T degenerates on

$$\partial_T[0,1] = \begin{cases} \{0,1\} & \text{if } b(\frac{1}{2}) \neq 1; \\ \{0, \frac{1}{2}, 1\} & \text{if } b(\frac{1}{2}) = 1\,. \end{cases}$$

3. Consider the canonical simplex K_d of $\mathbb{R}^d$, i.e.

$$K_d = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i \geq 0 \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\},$$

and the canonical projection[2] T_d on $C(K_d)$

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i) \quad (f \in C(K_d),\ x \in K_d), \tag{14}$$

where $v_0 := (0, \dots, 0), v_1 := (1, 0, \dots, 0), \dots, v_d := (0, \dots, 0, 1)$ are the vertices of K_d.

T_d fixes the functions $pr_1, \dots, pr_d$, and

$$W_{T_d}(u)(x) = \frac{1}{2} \sum_{i=1}^{d} x_i (1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1 \leq i < j \leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \tag{15}$$

$(u \in C^2(K_d),\ x = (x_1, \dots, x_d) \in K_d)$.

The coefficients of W_{T_d} vanish on the vertices of the simplex.

[2] T is a positive linear operator such that $T \circ T = T$.

4. Let $S : C(K_d) \longrightarrow C(K_d)$ be the Markov operator defined, for every $f \in C(K_d)$ and $x \in K_d)$, by

$$S(f)(x) = \begin{cases} \left(1 - \dfrac{x_1}{1 - \sum\limits_{i=2}^{d} x_i}\right) f(0, x_2, \dots, x_d) \\ + \dfrac{x_1}{1 - \sum\limits_{i=2}^{d} x_i} f\left(1 - \sum\limits_{i=2}^{d} x_i, x_2, \dots, x_d\right) & \text{if } \sum\limits_{i=2}^{d} x_i \neq 1; \\ f(0, x_2, \dots, x_d) & \text{if } \sum\limits_{i=2}^{d} x_i = 1 \end{cases} \tag{16}$$

S preserves the coordinate functions, and

$$W_S(u)(x) = \frac{1}{2} x_1 \left(1 - \sum_{i=1}^{d} x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x) \quad (u \in C^2(K_d), x \in K_d). \tag{17}$$

Note that W_S degenerates on the faces $\{x = (x_1, \dots, x_d) \in K_d \mid x_1 = 0\}$ and $\left\{x = (x_1, \dots, x_d) \in K_d \mid \sum\limits_{i=1}^{d} x_i = 1\right\}$.

Examples of W_T

5. Further examples can be obtained by means of tensor products of Markov operators (this is the case, for instance, of hypercubes) or by means of convex convolution products of Markov operators.

6. Suppose that ∂K is an ellipsoid, i.e. for $d \geq 2$,

$$K = \left\{ x \in \mathbb{R}^d \,:\, Q(x - \overline{x}) = \sum_{i,j=1}^{d} r_{ij}(x_i - \overline{x}_i)(x_j - \overline{x}_j) \leq 1 \right\}, \tag{18}$$

where $(r_{ij})_{i,j=1,\ldots,d}$ is a real symmetric and positive-definite matrix and $\overline{x} \in \mathbb{R}^d$.

Let L be a strictly elliptic differential operator of the form

$$L(u)(x) = \sum_{i,j=1}^{d} c_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \quad (u \in C^2(\mathrm{int}(K)), x \in \mathrm{int}(K)) \tag{19}$$

associated with a real symmetric and positive matrix $(c_{ij})_{1 \leq i,j \leq d}$ and denote by $T_L : C(K) \to C(K)$ the Poisson operator associated with L.

Thus, for every $f \in C(K)$, $T_L(f)$ denotes the unique solution to the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on } \mathrm{int}(K), \quad u \in C(K) \cap C^2(\mathrm{int}(K)); \\ u = f & \text{on } \partial K. \end{cases} \tag{20}$$

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Examples of W_T

T_L is a Markov operator preserving the coordinate functions. Moreover, under the above assumptions, the differential operator W_{T_L} associated to T_L, briefly W_L, is given by

$$W_L(f)(x) = \begin{cases} \dfrac{1 - Q(x - \overline{x})}{2\sum\limits_{i,j=1}^{d} r_{ij} c_{ij}} L(u)(x) & \text{if } x \in \operatorname{int}(K); \\ 0 & \text{if } x \in \partial K \end{cases} \tag{21}$$

$(u \in C^2(K),\ x \in K)$.

W_L degenerates on the ellipsoid ∂K.

Examples of W_T

For instance, if K is the closed ball (with respect to $\|\cdot\|_2$) of center the origin of $\mathbb{R}^d$ and radius 1 and $L = \Delta$, then

$$T_\Delta(f)(x) = \begin{cases} \dfrac{1-\|x\|_2^2}{\sigma_d} \displaystyle\int_{\partial K} \dfrac{f(z)}{\|z-x\|_2^d} d\sigma(z) & \text{if } \|x\|_2 \leq 1; \\ f(x) & \text{if } \|x\|_2 = 1 \end{cases} \tag{22}$$

($f \in C(K)$, $x \in K$), where σ_d and σ denote the surface area of the unit sphere in $\mathbb{R}^d$ and the surface measure on ∂K, resp.. In this case

$$W_\Delta(f)(x) = \begin{cases} \dfrac{1-\|x\|_2^2}{2d} \Delta(u)(x) & \text{if } \|x\|_2 < 1; \\ 0 & \text{if } \|x\|_2 = 1 \end{cases} \tag{23}$$

($u \in C^2(K)$, $x \in K$).

Bernstein-Schnabl operators associated with T

Given the Markov operator T fixed before, we can construct the following sequence of linear operators: for every $n \geq 1$, $f \in C(K)$ and $x \in K$,

$$B_n(f)(x) = \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n), \tag{24}$$

where $(\tilde{\mu}_x^T)_{x \in K}$ is the family of probability Borel measures on K uniquely associated with T via the Riesz representation theorem, i.e. for every $f \in C(K)$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T. \tag{25}$$

We call B_n the n-th Bernstein-Schnabl operator associated with T.

- B_n is a positive linear operator from $C(K)$ into $C(K)$;
- $B_n(\mathbf{1}) = \mathbf{1} \Longrightarrow \|B_n\| = 1$ for every $n \geq 1$;
- $B_1 = T$;
- If $K = [0,1]$ and T is

$$T(f)(x) = xf(1) + (1-x)f(0) \quad (f \in C([0,1]), x \in [0,1]), \tag{26}$$

then B_n's turn into the classical Bernstein operators on $[0,1]$.

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Properties of B_n's

- $(B_n)_{n\geq 1}$ is a positive approximation process on $C(K)$: for every $f \in C(K)$,

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on } K. \tag{27}$$

- if $m \geq 1$, denoting by $P_m(K)$ the space of (restriction to K of all) polynomials of degree at most m, then

$$B_n(P_m(K)) \subset P_m(K) \tag{28}$$

for every $n, m \geq 1$, provided that

$$T(P_m(K)) \subset P_m(K) \quad \text{for every } m \geq 1\,. \tag{29}$$

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- For every $u \in C^2(K)$,

$$\lim_{n\to\infty} n(B_n(u) - u) = W_T(u) \quad \text{uniformly on } K. \tag{30}$$

Thanks to (28) and (30), we may apply a generation and approximation result [Altomare, Cappelletti Montano, L., Rașa, 2014] getting the following result.

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Main result

Main result (Altomare, Cappelletti Montano, L., Rașa, 2014)

Let K be a convex compact subset of $\mathbb{R}^d$, $d \geq 1$, having non-empty interior and consider a Markov operator T on $C(K)$ such that

$$T(h) = h \quad \text{for every } h \in \{pr_1, \ldots, pr_d\}. \tag{Hp1}$$

Furthermore, assume that

$$T(P_m(K)) \subset P_m(K) \qquad \text{for every } m \geq 2. \tag{Hp2}$$

Then the operator $(W_T, C^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(S(t))_{t\geq 0}$ on $C(K)$ such that, if $t \geq 0$ and $(k_n)_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k_n/n = t$,

$$S(t)(f) = \lim_{n\to\infty} B_n^{k_n}(f) \quad \text{uniformly on } K \tag{$\star$}$$

for every $f \in C(K)$.

Applications of the approximation formula $(\star)$

Proposition

Let U be a closed (with respect to the uniform norm) subset of $C(K)$. Then

$$B_n(U) \subset U \Longrightarrow S(t)(U) \subset U \quad \text{for every } t \geq 0. \tag{31}$$

Then, considering

$$\begin{cases} \frac{du}{dt}(t) = A_T(u(t)) & t \geq 0; \\ \\ u(0) = u_0 & u_0 \in D(A_T), \end{cases} \tag{32}$$

if $u_0 \in U \cap D(A_T)$, then $u(t) \in U$, u being the unique solution of (33).

- $U = P_m(K)$;
- $U = \mathrm{Lip}(M, \alpha) \quad (M \geq 0, \alpha \in]0,1])$ under the additional assumtion

$$T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1) \tag{33}$$

  (for instance the canonical projection T_d of K_d satisfies (34) endowed with the l_1-metric);
- the set U of all continuous convex functions on $[0,1]$;
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- Consider the Poisson operator $T_L : C(K) \to C(K)$, K being an ellipsoid of the form considered before, associated with a stricltly elliptic differential operator L with constant coefficients. Then

$$\lim_{t\to+\infty} S(t)(f) = T_L(f) \quad \text{uniformly on } K, \text{ for every } f \in C(K). \tag{34}$$

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- Other fields where the representation formula $(\star)$ can be usefully applied are concerned with

- Markov processes (probabilistic applications);
- estimates of the quantity

$$\|S(t)(f) - B_n^{k_n}(f)\|_\infty \quad (f \in C(K)) \tag{36}$$

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Complete differential operators associated with T

Coming back to

$$V_T(u) = W_T(u) + \sum_{i=1}^{d} \beta_i \frac{\partial u}{\partial x_i} + \gamma u \quad (u \in C^2(K)), \tag{37}$$

we are able to establish some conditions under which the problem stated at the beginning may be solved. In particular

- we assume that, for every $i = 1, \ldots, d$,

$$\beta_i \text{ is the restriction to } K \text{ of a polynomial of degree at most 1;} \tag{38}$$

- we define new operators M_n by modifying the Bernstein-Schnabl operators whose iterates approximate the semigroup pre-generated by W_T. Operators M_n defined on $C(K)$ as

$$M_n(f) = B_n\left(\left(\mathbf{1} + \frac{\gamma}{n}\right) \times \left(f \circ \left(id_K + \frac{\beta}{n}\right)\right)\right) \tag{39}$$

under the requirement that there exists $n_0 \geq 1$ such that, for every $x \in K$ and $n \geq n_0$,

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