Γ-convergence for integral functionals depending on vector fields

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Workshop
”Classical and new methods in Calculus of Variations and PDEs”
Università di Salerno
Salerno, January 10, 2019

Content

Vector fields

We assume that $X_1, \ldots, X_m$ are Lipschitz continuous vector fields (v.f.) on an open set $\Omega \subset \mathbb{R}^n$, i.e. $X_j = (c_{j1}, \ldots, c_{jn})$, with $c_{ji} \in Lip(\Omega)$ for $j = 1, \ldots, m$, $i = 1, \ldots, n$. We identify

$$X_j = \sum_{i=1}^{n} c_{ji}(x)\partial_i \,.$$

Moreover, we define as *X-gradient*

$$X := (X_1, \ldots, X_m)$$

and as *coefficient matrix of X-gradient* the $m \times n$ matrix

$$C(x) = [c_{ji}(x)]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \,.$$

Vector fields

Definition

We say that $X = (X_1, \dots, X_m)$ *satisfies the full rank condition (FRC) on an open set* $\Omega \subset \mathbb{R}^n$*, if there exists a closed set* $\mathcal{N}_X \subset \Omega$ *such that* $\mathcal{L}^n(N_X) = 0$ *and, for each* $x \in \Omega_X := \Omega \setminus \mathcal{N}_X$, $X_1(x), \dots, X_m(x)$ *are linearly independent as vectors of* $\mathbb{R}^n$*.*

Rmk. Notice that if $X = (X_1, \dots, X_m)$ satisfies (FRC), then $m \leq n$.

Vector fields

Examples of relevant vector fields.

(i) (Euclidean gradient) Let $X = (X_1, \dots, X_n) = \nabla := (\partial_{x_1}, \dots, \partial_{x_n})$.

(ii) (Grushin v.f.) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^2$ defined as

$$X_1(x) := \partial_{x_1}, \quad X_2(x) := x_1\, \partial_{x_2} \text{ if } x = (x_1, x_2) \in \mathbb{R}^2 .$$

(iii) (Heisenberg v.f.) Let $X = (X_1, X_2)$ be the v.f. on $\mathbb{R}^3$ defined as

$$X_1(x) := \partial_{x_1} - \frac{x_2}{2}\partial_{x_3},\ X_2(x) := \partial_{x_2} + \frac{x_1}{2}\partial_{x_3} \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

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Vector fields

Rmk. Notice that all three families of v.f. satisfy (FRC) respectively in $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^2$ and $\Omega = \mathbb{R}^3$.

Functionals depending on vector fields

We will deal with integral functionals $F: L^p(\Omega) \to [0, \infty]$, $1 < p < \infty$, of the form

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in C^1(\Omega) \\ \infty & \text{if } u \in L^p(\Omega) \setminus C^1(\Omega) \end{cases},$$

with *integrand function* $f: \Omega \times \mathbb{R}^m \to [0, \infty]$ in the class $I_{m,p}(\Omega, c_0, c_1)$, composed by functions verifying the following assumptions:

(I_1) for every $\eta \in \mathbb{R}^m$, the function $f(\cdot, \eta): \Omega \to [0, \infty]$ is Borel measurable on Ω;

(I_2) for a.e. $x \in \Omega$, the function $f(x, \cdot): \mathbb{R}^m \to [0, \infty)$ is convex;

(I_3) there exist constants $c_1 \geq c_0 > 0$ such that

$$c_0 \, |\eta|^p \leq f(x, \eta) \leq c_1 \left(|\eta|^p + 1\right),$$

for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$.

Functionals depending on vector fields

If $f \in I_{m,p}(\Omega, c_0, c_1)$, $u \in C^1(\Omega)$, and

$$F(u) = \int_\Omega f(x, Xu(x))dx\,,$$

then

$$F(u) = \int_\Omega f_e(x, Du(x))dx\,,$$

with

$$f_e(x,\xi) := f(x, C(x)\xi) \text{ if } \xi \in \mathbb{R}^n\,.$$

f_e will be called Euclidean integrand associated to F.

Functionals depending on vector fields

Rmk. One can prove that the opposite representation may not hold.

Examples of functionals depending on vector fields.

Let $f(x,\eta) = |\eta|^2$. Then

(i) (Grushin v.f.)

$$F(u) = \int_\Omega f(x, Xu)\,dx = \int_\Omega \left(\partial_1 u^2 + x_1^2 \partial_2 u^2\right)\,dx \text{ if } u \in C^1(\Omega)\,;$$

(ii) (Heisenberg v.f.)

$$F(u) = \int_\Omega f(x, Xu)\,dx = \int_\Omega \Big((\partial_1 u - \frac{x_2}{2}\partial_3 u)^2 + (\partial_2 u + \frac{x_1}{2}\partial_3 u)^2\Big)\,dx \text{ if } u \in C^1(\Omega)\,.$$

Functionals depending on vector fields

Rmk. Observe that the previous functionals are not coercive w.r.t. the Euclidean gradient, that is, coercivity condition

$$f_e(x,\xi) \geq c_0 \, |\xi|^2 \text{ a.e. } x \in \Omega, \, \forall \xi \in \mathbb{R}^n ,$$

for a suitable constant $c_0 > 0$, may fail.

Functionals depending on vector fields

A good feature of functionals depending on v.f. is that they enable to deal with degenerate functionals depending on the Euclidean gradient.

Content

Γ-convergence

Γ-compactness' problem. Let $X = (X_1, \dots, X_m)$ be a given family of Lipschitz v.f. on a bounded open set $\Omega \subset \mathbb{R}^n$ and let $F_h : L^p(\Omega) \to [0, \infty]$,

$$F_h(u) := \begin{cases} \int_\Omega f_h(x, Xu(x))dx & \text{if } u \in C^1(\Omega) \\ \infty & \text{if } u \in L^p(\Omega) \setminus C^1(\Omega) \end{cases},$$

with $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$. Are there a function $f \in I_{m,p}(\Omega, c_0, c_1)$ and a functional $F : L^p(\Omega) \to [0, \infty]$ such that, up to a subsequence,

- $F = \Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h$,
- $F(u) = \int_\Omega f(x, Xu(x))\, dx$ for each $u \in C^1(\Omega)$?

Moreover we would like to characterize

$$\mathrm{dom} F := \{u \in L^p(\Omega) : F(u) < \infty\}.$$

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Relaxation of functionals depending on v.f.

Assume that $f_h = f \in I_{m,p}(\Omega, c_0, c_1)$ for each h. Then it is well-known that

$$\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h(u)$$

$$= \bar{F}(u) := \inf\left\{ \liminf_{h\to\infty} F(u_h) : (u_h)_h \subset L^p(\Omega), u_h \to u \text{ in } L^p(\Omega) \right\}.$$

$\bar{F}$ is called the relaxed functional of F w.r.t. the L^p-topology.

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Sobolev spaces depending on v.f.

For $1 \leq p \leq \infty$ we set

$$W^{1,p}_X(\Omega) := \{u \in L^p(\Omega) \quad : \quad X_j u \in L^p(\Omega) \quad \text{for } j = 1, \dots, m\}$$

Rmk. Since vector fields X_j have Lipschitz continuous coefficients, it is immediate that

$$W^{1,p}(\Omega) \subset W^{1,p}_X(\Omega) \quad \forall\, p \in [1, \infty]\,,$$

and

$$Xu(x) = C(x)\, Du(x) \quad \text{for a.e. } x \in \Omega\,,$$

where $W^{1,p}(\Omega)$ denotes the classical Sobolev space, or, equivalently, the space $W^{1,p}_X(\Omega)$ associated to $X = \nabla := (\partial_{x_1}, \dots, \partial_{x_n})$. Moreover it is easy to see that inclusion can be strict.

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Characterization of the relaxed functional

Theorem ([Franchi-Serapioni-S.C., 1996])

Let $X = (X_1, \ldots, X_m)$ be a given family of v.f. on a open set $\Omega \subset \mathbb{R}^n$ and let $p > 1$.
Then

- (i) $\mathrm{dom}\, \bar{F} = W^{1,p}_X(\Omega)$;
- (ii) $\bar{F}(u) = \int_\Omega f(x, Xu(x))\, dx$ for every $u \in W^{1,p}_X(\Omega)$.

The Γ-compactness result

We are able to prove the Γ-compactness result only for three subclasses $J_i \subset I_{m,p}(\Omega, c_0, c_1)$ $(i = 1, 2, 3)$.

- J_1 is composed by integrand functions $f \in I_{m,2}(\Omega, c_0, c_1)$ which are quadratic forms w.r.t. η, that is,

$$f(x, \eta) = \langle a(x)\eta, \eta\rangle = \sum_{i,j=1}^{m} a_{ij}(x)\eta_i\eta_j \quad \text{a.e. } x \in \Omega, \forall\, \eta \in \mathbb{R}^m,$$

  with $a(x) = [a_{ij}(x)]$ $m \times m$ symmetric matrix .
- the subclass $J_2 = I_{n,p}(\Omega, c_0, c_1)$.
- the subclass J_3 is composed by $f \in I_{m,p}(\Omega, c_0, c_1)$ such that $f = f(\eta)$, that is, f independent of x;

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Theorem (Maione-Pinamonti-S.C., 2018)

Let $(f_h)_h \subset J_i$ ($i = 1, 2, 3$) and let $(F_h)_h$ be the associated sequence of integral functionals on $L^p(\Omega)$, $1 < p < \infty$. Assume that :

- *in the case of subclasses J_i with $i = 1, 2$, $X = (X_1, \dots, X_m)$ satisfies (FRC) on $\Omega \subset \mathbb{R}^n$;*
- *in the case of subclass J_3, $X = (X_1, \dots, X_m)$ induces a Carnot group structure on $\mathbb{R}^n$.*

Then, up to a subsequence, there is $f \in J_i$ such that, for each $u \in L^p(\Omega)$

$$\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h(u) = \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

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Proof outline

We first localize functionals by considering functionals defined on $L^p(\Omega) \times \mathcal{A}$, where $\mathcal{A}$ denotes the class of open sets of Ω,that is,

$$F_h(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in C^1(A) \\ \infty & \textit{otherwise} \end{cases}.$$

Moreover, WLOG, by well-known resuts, we can study the Γ-convergence of the relaxed functionals of F_h,

$$F_h^*(u, A) := \bar{F}_h(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W_X^{1,p}(A) \\ \infty & \textit{otherwise} \end{cases}.$$

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Proof outline

Proof's strategy consists in two steps.

1st step. By applying classical results from the Euclidean setting (contained in G. Dal Maso, *An Introduction to Γ-convergence*, 1993) we prove that : let $(F_h)_h$ be a sequence of integral functionals on $L^p(\Omega) \times \mathcal{A}$, $1 < p < \infty$, of the form

$$F_h(u, A) := \begin{cases} \int_A f_{h,e}(x, Du(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,1}_{\mathrm{loc}}(A) \\ +\infty & \text{otherwise} \end{cases},$$

where

$$f_{h,e}(x, \xi) := f_h(x, C(x)\xi) \quad x \in \Omega,\ \xi \in \mathbb{R}^n .$$

Proof outline

Then, up to a subsequence, there exists
$F: L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that

$$F(\cdot, A) = \Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}, \qquad (1)$$

and F can be represented by an integral form on $W^{1,p}(A)$ by means of an **Euclidean integrand function**, that is,

$$F(u, A) := \int_A f_e(x, Du(x))\, dx \qquad (2)$$

for every $A \in \mathcal{A}$, for every $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}(A)$for a suitable Borel function $f_e: \Omega \times \mathbb{R}^n \to [0, \infty]$.

Proof outline

2nd step. We prove that subclasses J_i ($i = 1, 2, 3$) satisfy the following closure property w.r.t. Γ-convergence: assume that $(f_h)_h \subset J_i$ and (1) and (2) hold, then F

$$\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h^*(u) = \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W_X^{1,p}(\Omega) \\ \infty & \text{otherwise} \end{cases} .$$

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Open problems

(i) Γ-compactness result for a sequence of integrands $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$?

(ii) Convergence of functionals's minimizers?
Previous results on H-convergence for linear PDEs depending on v.f. generating a Carnot group structure by [Franchi,Tchou, Tesi, 2006] and [Baldi, Franchi, Tchou, Tesi, 2010]

(iii) Homogeneization for functionals depending on vector fields? Previous results for linear PDEs depending on v.f. generating a Carnot group structure by [Franchi, Tesi, 2002] and [Franchi, Gutierrez, van Nguyen, 2005].

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