Fractional perimeters and related Allen-Cahn PDEs

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Classical and new methods in Calculus of Variations and PDEs

Classical Perimeters

Motivation: Local interactions, e.g.

$$\min\Big\{P(F,U),\ (\mathbb{R}^n \setminus U)\cap F = (\mathbb{R}^n \setminus U)\cap E\Big\}, E, U \subset \mathbb{R}^n \textit{ given},$$

Here

$$P(F,U) = \sup\Big\{\int_F \operatorname{div}\varphi dx,\ \varphi \in [C_c^1(U)]^n\Big\}$$

is the surface measure of the part of ∂F in U.

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Aim of the talk: Fractional perimeters in the euclidean space

Fractional perimeters in the Gaussian space

The isoperimetric problem

Applications to some semilinear (nonlocal) elliptic equations

Fractional Perimeters in the euclidean space

Motivation: Fractional diffusion and nonlocal interactions. For $0 < s < 1$, set

$$P_s(E) = \frac{1}{2}\mathcal{J}_s(E, E^c),$$

where

$$\mathcal{J}_s(A, B) = \iint \frac{1}{|x-y|^{n+s}} \chi_A(x)\chi_B(y)\, dxdy$$

Remark $P_s(E) \approx \|(-\Delta)^{s/4}\chi_E\|^2_{L^2}$.

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References: Savin, Ann. of Math. 2009
Caffarelli-Roquejoffre-Savin, CPAM 2010
Caffarelli-Valdinoci, Calc. Var. 2011

The nonlocal minimisation problem

The analogue of the above minimisation problem reads

$$\min\Big\{ P_s(F,U),\ (\mathbb{R}^n \setminus U) \cap F = (\mathbb{R}^n \setminus U) \cap E \Big\}, E, U \subset \mathbb{R}^n \textit{ given},$$

where

$$\begin{aligned} P_s(F;U)) = {} & \mathcal{I}_s(F \cap U, F^c \cap U) \\ & + \mathcal{I}_s(F \cap U, F^c \cap U^c) \\ & + \mathcal{I}_s(F \cap U^c, F^c \cap U) \end{aligned}$$

The extension approach for the fractional Laplacean

Thm Let $u \in \mathrm{dom}((-\Delta)^\sigma)$ and v a solution of the extension problem

$$\begin{cases} \Delta v + \dfrac{1-2\sigma}{y}\partial_y v + \partial_y^2 v = 0 & \text{on } \mathbb{R}^n \times (0, +\infty) \\ v(x,0) = u & \text{on } \mathbb{R}^n, \end{cases}$$

Then,

$$-\lim_{y \to 0^+} y^{1-2\sigma}\partial_y v(x,y) = (-\Delta)^\sigma u(x) \qquad \textit{in } L^2.$$

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References: Caffarelli-Silvestre CPDE 2007
Stinga-Torrea, CPDE 2010

The variational formulation for the fractional perimeter

Theorem

$$P_s(E) = \inf\Big\{ \int_{\mathbb{R}^n\times\mathbb{R}^+} \left(|\nabla v|^2 + |\partial_y v|^2\right) y^{1-s} dxdy : \\ v \in H^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^+),\ v(\cdot,0) = \chi_E(\cdot)\Big\}.$$

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Remark If $P_s(E) < \infty$, then a minimiser v_E always exists in the weighted Sobolev space

$$H^1(\mathbb{R}^n \times \mathbb{R}^+, dx \otimes y^{1-s} dy)$$

and χ_E belongs to the fractional Sobolev space $H^{s/2}(\mathbb{R}^n)$. Indeed,

$$P_s(E) = \frac{1}{2} \|\chi_E\|^2_{H^{s/2}}.$$

Perimeters in the Gaussian case

Given the Gaussian measure $\gamma = \mathcal{N}(0, Q)$ on $\mathbb{R}^n$, let

$$P_\gamma(E) = \sup\Big\{ \int_{\mathbb{R}^n} \chi_E \, \mathrm{div}_\gamma \varphi d\gamma, \ \varphi \in [C^1_c(\mathbb{R}^n)]^n \Big\}$$

where $\mathrm{div}_\gamma \varphi = \sum_{j=1}^{n} \partial_j \varphi_j - (Qx)_j \varphi_j$.

The infinite dimensional case

Remark The definition of $P_{\gamma,s}$ and all that follows can be rephrased in the abstract Wiener space (X, γ, H), where X is a separable Banach space, γ is a Gaussian measure and H is the Cameron-Martin space.
Here $\gamma = \mathcal{N}(0, Q)$, $Q = RR^* : X^* \to X$ is the covariance operator, $R^* : X^* \hookrightarrow L^2_\gamma(X)$ is the canonical embedding, H is the R-image of the closure of R^*X^* in $L^2_\gamma(X)$.

Warning: From now on I work in the Wiener space X, but X can be thought of as $\mathbb{R}^n$ with a (even standard) Gaussian measure γ.

The isoperimetric problem in the Gaussian space

Theorem For any $m \in (0,1)$ there exists a set $E_m \subset X$ which solves the isoperimetric problem

$$\min\Big\{P_\gamma(E) : \ E \subset X, \ \gamma(E) = m\Big\}.$$

Moreover, the set E_m is necessarily a half-space.

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Ehrhard symmetrisation:

$$\phi(t) = \int_{-\infty}^{t} e^{-s^2/2} ds, \quad I \subset \mathbb{R} \quad \Rightarrow \; I^* = (-\infty, \phi^{-1}(\gamma_1(I))$$

$$|h| = 1, \; x' = x - \langle h, x \rangle h, \; x = x' + th \; with \; t \in \mathbb{R}$$

$$u_h^*(x' + th) = \sup\Big\{c \in \mathbb{R}: \; t \in \{u(x', \cdot) > c\}^*\Big\}, \chi_{E_h^*} = (\chi_E)_h^*.$$

The Gaussian fractional perimeter

$$P_{\gamma,s}(E) = \frac{1}{2}[\chi_E]^2_{H^{s/2}_\gamma}$$

where

$$[\chi_E]^2_{H^{s/2}_\gamma} = \int\int \frac{1}{|x-y|^{n+s}} \chi_E(x)\chi_{E^c}(y)\, d\gamma(x)d\gamma(y)$$

and

$$P_{\gamma,s}(E) \approx \|(-\Delta_\gamma)^{s/4}\chi_E\|^2_{L^2_\gamma}.$$

where Δ_γ is the Ornstein-Uhlenbeck operator .

The extension approach in the Gaussian case

Define the Dirichlet form

$$\mathcal{E}(u,v) = \int_X \langle \nabla u, \nabla v \rangle d\gamma$$

and Δ_γ through

$$\int_X (-\Delta_\gamma u) v d\gamma = \mathcal{E}(u,v) \qquad \forall\, v \in \mathrm{dom}(\mathcal{E})$$

Thm Let $u \in \mathrm{dom}((-\Delta_\gamma)^\sigma)$ and v a solution of the extension problem

$$\begin{cases} \Delta_\gamma v + \dfrac{1-2\sigma}{y}\partial_y v + \partial_y^2 v = 0 & \text{on } X \times (0,+\infty) \\ v(x,0) = u & \text{on } X, \end{cases}$$

Then,

$$-\lim_{y\to 0^+} y^{1-2\sigma}\partial_y v(x,y) = (-\Delta_\gamma)^\sigma u(x) \qquad \textit{in } L^2_\gamma.$$

The variational extension approach

$$P_{\gamma,s}(E) = \inf\Big\{ \int_{X\times\mathbb{R}^+} \left(|\nabla v|^2 + |\partial_y v|^2\right) y^{1-s} d\gamma(x) dy : \\ v \in H^1_{\mathrm{loc}}(X\times\mathbb{R}^+),\ v(\cdot,0) = \chi_E(\cdot) \Big\}.$$

Remark If $P_{\gamma,s}(E) < \infty$, then a minimiser v_E always exists in the weighted Gussian Sobolev space

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Moreover, the set E_m is necessarily a half-space.
Remark In the Wiener space, the isoperimetric sets are $E_m = \{\hat{h} < c\}$ for some $h = R\hat{h} \in H$ and $c \in \mathbb{R}$.

Idea of the proof

By the extension approach, we deal with an isoperimetric problem with a mixed measure.

$$P_{\gamma,s}(E) = \inf\Big\{ J_1(v) + J_2(v) : \ v(\cdot, 0) = \chi_E(\cdot) \Big\},$$

where

$$J_1(v) := \int_{X\times\mathbb{R}^+} |\nabla v|^2 \, y^{1-s} d\gamma(x) dy,$$
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Then, $J_1(v_h^*) \leq J_1(v)$ and $J_2(v_h^*) \leq J_2(v)$.

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The second inequality is less standard because one deals with $\partial_y v$ and v is rearranged wrto x.

A fractional Allen-Cahn equation

The above described extension technique can be used to prove symmetry properties of solutions of the following fractional semilinear elliptic equation: $(-\Delta_\gamma)^s u = f(u)$.

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$$v(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta_\gamma}((-\Delta_\gamma)^s u)(x) e^{-y^2/4t} \frac{dt}{t^{1-s}}$$

be the solution of the extension problem

$$\begin{cases} \Delta_\gamma v + \dfrac{1-2s}{y} \partial_y v + \partial_y^2 v = 0 & \text{on } X \times (0,+\infty) \\ -\lim\limits_{y\to 0^+} y^{1-2s} \partial_y v(x,y) = f(u) & \text{on } X, \\ v(x,0) = u & \text{on } X, \end{cases}$$

Main result

Theorem (Novaga-P.-Sire) Let $u \in C^2(X) \cap L^\infty(X)$, $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz and v the solution of the extension problem. If

$$(\textit{monotonocity}) \qquad \inf_{x \in X} \langle \nabla u(x), w \rangle > 0$$

for some w and v is C^2 and continuous up to the boundary $y = 0$
Then u is 1D, i.e., $\exists U : \mathbb{R} \to \mathbb{R}$ and $\omega \in X^*$ such that

$$u(x) = U(\langle \omega, x \rangle) \qquad \textit{for all } x \in X.$$

The story begins

De Giorgi's conjecture (1978) Let $u : \mathbb{R}^n \to (-1, 1)$ be a solution of

$$\Delta u = u^3 - u$$

in $\mathbb{R}^n$ with $\partial_{x_n} u > 0$ everywhere. Then the level sets $\{u = c\}$ are hyperplanes (at least for $n \leq 8$).

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Related to the Modica-Mortola approximation: $\Omega \subset \mathbb{R}^n$ smooth,

$$\Gamma - \lim_{\varepsilon \to 0} \int_\Omega \varepsilon |Du|^2 + \frac{1}{\varepsilon}(1-u^2)^2 =$$

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Remark del Pino, Kowalczyk, Wei (2008): counterexample for $n \geq 9$.

Previous results

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1997 Ghoussoub–Gui ($n = 2$), 2000 Ambrosio–Cabré ($n = 3$), 2001 Alberti–Ambrosio–Cabré (more general nonlinearities)

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2005 Cabré–Solà-Morales; 2007 Sire–Valdinoci; 2008 Cabré–Sire
The local Gaussian case (in Wiener space) $-\Delta_\gamma u = f(u)$
2014 Cesaroni–Novaga–Valdinoci

Sketch of the proof

Extension technique:

$$\begin{cases} \Delta_\gamma v + \dfrac{1-2s}{y}\partial_y v + \partial_y^2 v = 0 & \text{on } X \times (0,+\infty) \\ -\lim\limits_{y\to 0^+} y^{1-2s}\partial_y v(x,y) = f(u) & \text{on } X, \\ v(x,0) = u & \text{on } X, \end{cases}$$

Strategy u monotone $\Rightarrow$ v monotone

$\Rightarrow \quad v(x,y) = V(\langle\omega,x\rangle,y) \Rightarrow \quad u(x) = V(\langle\omega,x\rangle,0).$

Sketch of the proof

Step 1: Regularity of v
Step 2: Representation formula for v gives monotonicity
Step 3: monotonicity gives $v(\cdot, y)$ is $1D$ for any y (a geometric Poincaré inequality on the level sets)
Step 4: (X Wiener) limit as the dimension goes to ∞.