An homogenization result with a Γ-limit absolutely continuous with respect to a singular measure

joint work by
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We present a Γ-convergence result in which the Γ-limit consists only of its Cantorian part and is fully described

- We consider the sequence of integral functionals obtained considering some very simple integrand of the CoV with respect to a sequence of measures μ_h
- The measures μ_h , while absolutely continuous with respect to the Lebesgue measure, are noncoercive and weakly-* converge to a singular measure μ
- The sequence of functionals, properly rescaled, does Γ-converge
- The Γ-limit can be still identified as an integral functional with respect to **another** singular measure ν
- The measure ν depends both on the measure μ and the integrand

One dimensional toy measures: binomial measures

- Let $\alpha \in (0,1)$. The binomial measure μ_α is the only measure of total unit mass on the interval $I = [0,1)$ that satisfies the following relation

$$\mu_\alpha(J_L) = \alpha \mu_\alpha(J)$$

where $J = [\frac{n}{2^k}, \frac{n+1}{2^k})$, $0 \leq n \leq 2^k - 1$ is any dyadic subinterval of I and $J_L = [\frac{2n}{2^{k+1}}, \frac{2n+1}{2^{k+1}})$ is the left half of J

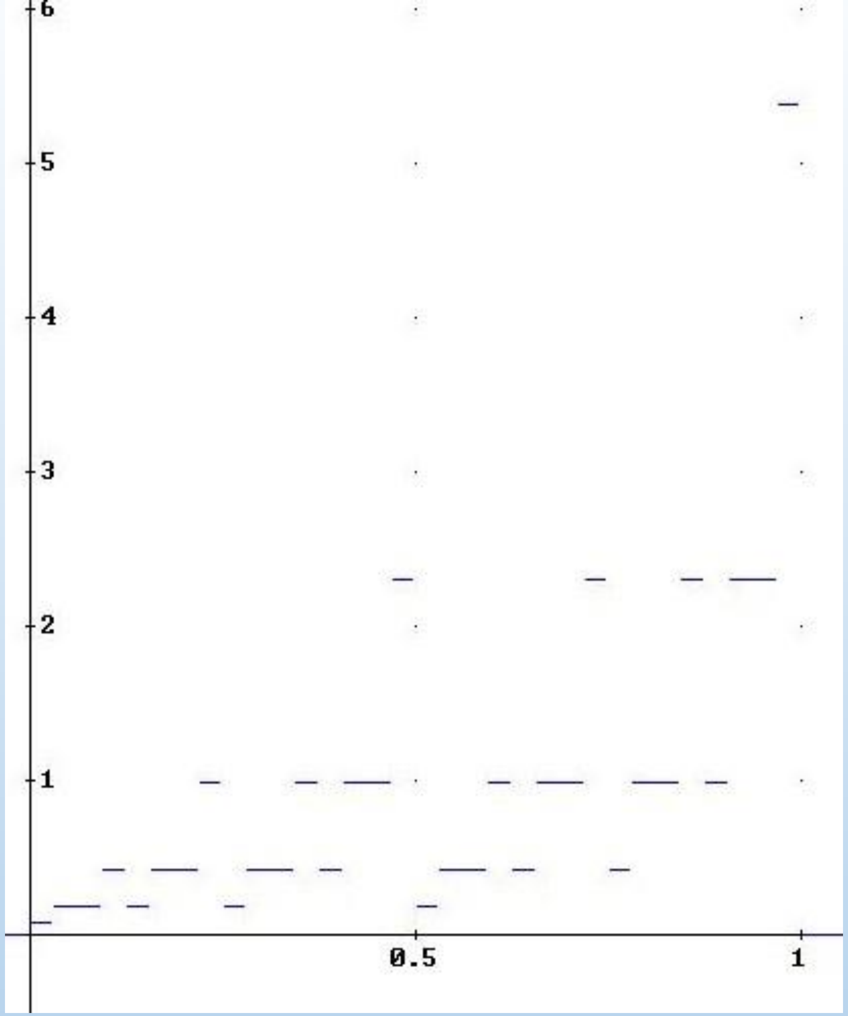
- The binomial measure μ_α is a continuous measure orthogonal with respect to the Lebesgue measure if $\alpha \neq \frac{1}{2}$

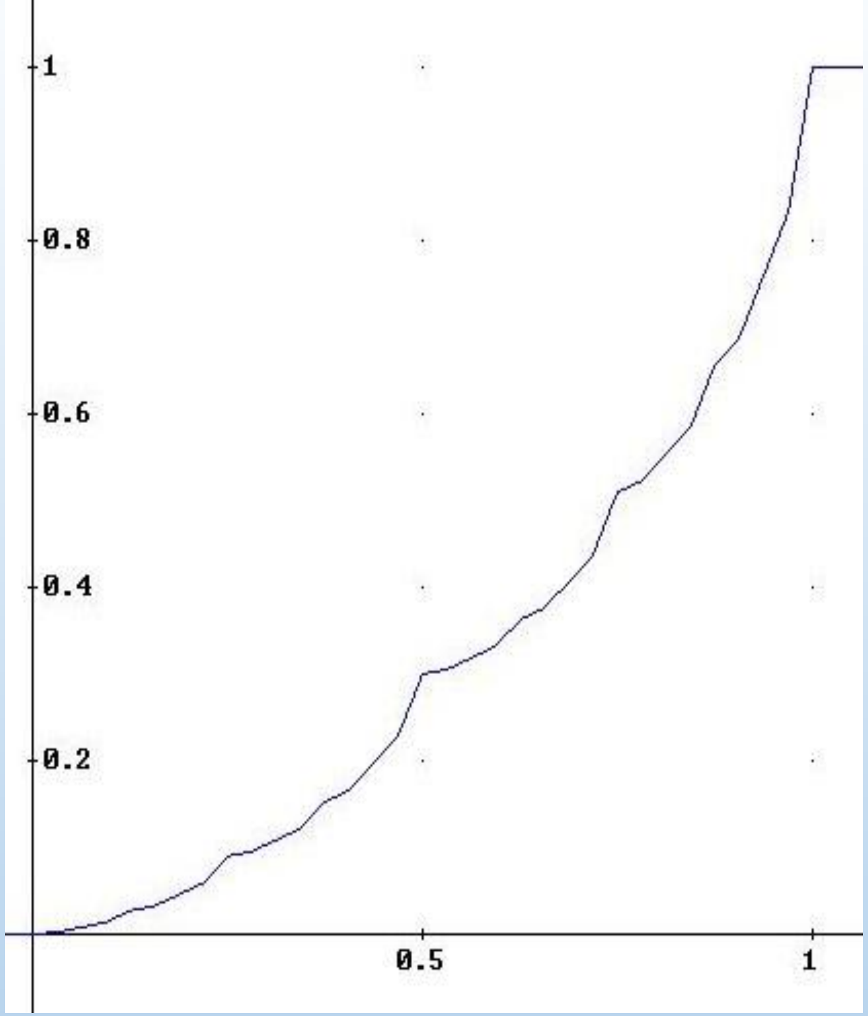
Approaching μ_α . Measures $\mu_{k,\alpha}$

The measure $\mu_{k,\alpha}$ is defined as the only measure on I such that

- $\mu_{k,\alpha}(J) = \mu_\alpha(J) \quad \forall\, J = [\frac{n}{2^k}, \frac{n+1}{2^k})$ dyadic interval "of level k"

- $\mu_{k,\alpha}$ is absolutely continuous with respect to the Lebesgue measure and has constant density on every dyadic interval "of level k"





On the left: example of density of $\mu_{k,\alpha}$. $\alpha = 0.3$, $k = 5$
On the right: graph of $\mu_{k,\alpha}([0,x])$, same parameters.

The toy functional E_k

- Let us fix α and consider the following functionals

$$E_k(u) = \int_0^1 |u'(x)|^2 d\mu_{k,\alpha}$$

- We want to minimize $E_k(u)$ among all functions in

$$\mathrm{V} = \{u \in W^{1,2}(I) | u(0) = 0, u(1) = 1\}$$

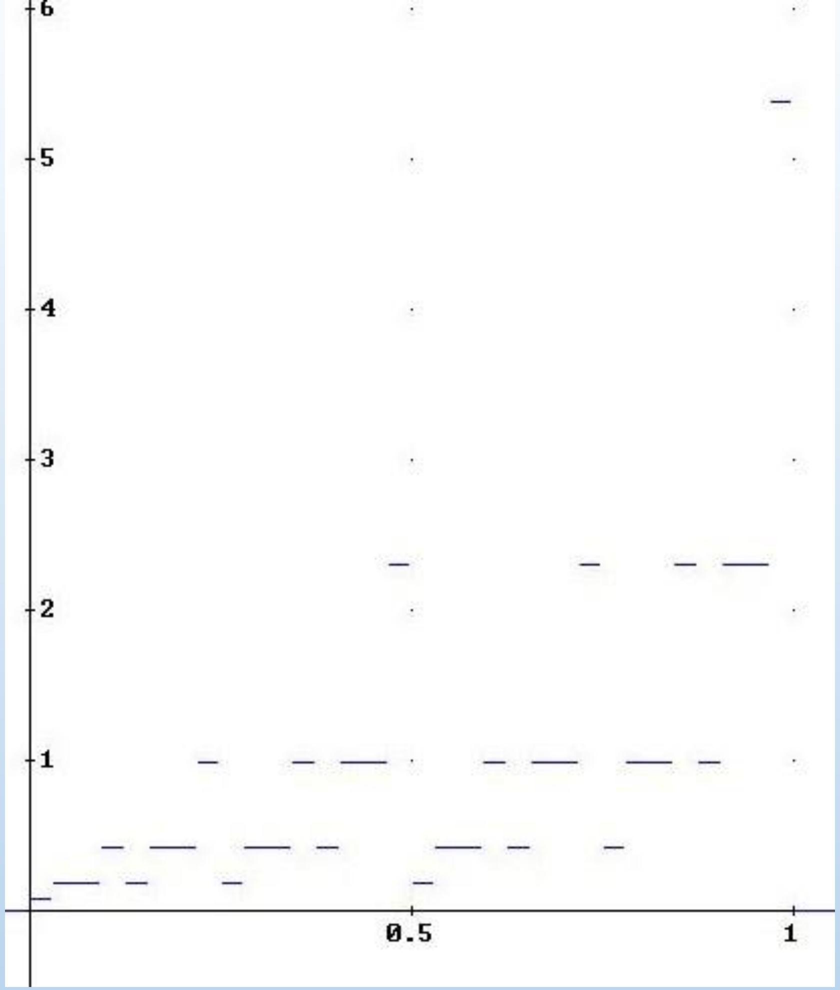
Lemma

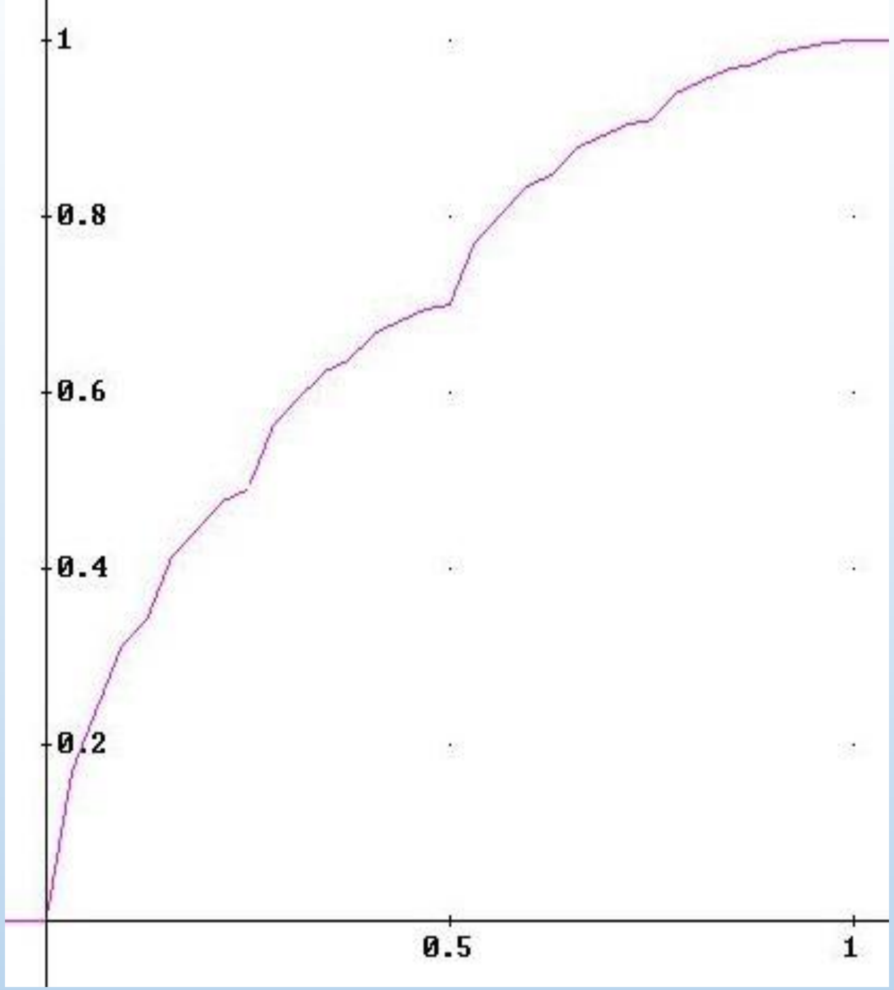
$$\min_{v \in V} E_k(v) = E_k(u_k) = \rho^k$$

where $\rho = 4\alpha(1-\alpha)$ and $u_k(x) = \mu_{k,1-\alpha}([0,x])$

Since $\mu_{k,1-\alpha}([0,x]) \longrightarrow \mu_{1-\alpha}([0,x])$ uniformly on I this suggests the idea $\mu_{1-\alpha}([0,x])$ can actually be the minimum of a variational problem.

On the other hand we have $\rho^k \longrightarrow 0$, therefore in order to correctly state our first result we have to properly rescale functionals E_k





On the left: density of $\mu_{k,\alpha}$, $\alpha = 0.3$, $k = 5$
On the right: graph of $u_k(x) = \mu_{k,1-\alpha}([0, x])$, same parameters.

Theorem 1

Let $\{F_k\}_k$ the sequence of functionals defined as follows

$$F_k(u) = \begin{cases} \rho^{-k} E_k(u) & u \in V \\ +\infty & otherwise\ in\ BV([0,1]) \end{cases}$$

Let us consider either L^1 or L^∞ norm on $BV([0,1])$. Then

$$F(u) = \Gamma^- \ (L^1) - \lim(\ F_k) = \Gamma^-(L^\infty) - \lim(\ F_k)$$

where

$$F(u) = \begin{cases} \int_0^1 \left| \frac{dDu}{d\mu_{1-\alpha}} \right|^2 d\mu_{1-\alpha} & if\ Du \ll \mu_{1-\alpha}\ and\ u(0) = 0, u(1) = 1 \\ +\infty & otherwise\ in\ BV([0,1]) \end{cases}$$

Case $1 < p < +\infty$

- Let us fix α and modify the functionals E_k as follows

$$E_k(u) = \int_0^1 |u'(x)|^p d\mu_{k,\alpha}$$

- We want to minimize $E_k(u)$ among all functions in

$$\mathrm{V} = \left\{u \in W^{1,p}(I) | u(0) = 0, u(1) = 1 \right\}$$

We can prove, in analogy with the case $p = 2$, the following

Lemma

$$\min_{v\in V} E_k(v) = E_k(u_k) = \left(\rho_p\right)^k$$

where $\rho_p = \frac{2^p\alpha(1-\alpha)}{\left(\alpha^{\frac{1}{p-1}}+(1-\alpha)^{\frac{1}{p-1}}\right)^{p-1}} = 2\alpha(2c)^{p-1}$, $c = c(\alpha,p) = \frac{(1-\alpha)^{\frac{1}{p-1}}}{\alpha^{\frac{1}{p-1}}+(1-\alpha)^{\frac{1}{p-1}}}$

and $u_k(x) = \mu_{k,c}([0,x])$. We have always $\rho_p < 1$ if $\alpha < \frac{1}{2}$

Moreover

$$\begin{cases} c\left(\frac{1}{2},p\right) = \frac{1}{2} \quad 1<p<+\infty, \qquad c(\alpha,2) = 1-\alpha \quad \forall\ 0<\alpha<1 \\ c(\alpha,p) \to 1^- \ as\ p \to 1^+, c(\alpha,p) \to \frac{1}{2}^+ \ as\ p \to +\infty \qquad if\ \alpha < \frac{1}{2} \\ c(\alpha,p) \to 0^+ \ as\ p \to 1^+, c(\alpha,p) \to \frac{1}{2}^- \ as\ p \to +\infty \qquad if\ \alpha > \frac{1}{2} \end{cases}$$

Theorem 1'

Let $\{F_k\}_k$ the sequence of functionals defined as follows

$$F_k(u) = \begin{cases} (\rho_p)^{-k} E_k(u) & u \in V \\ +\infty & otherwise\ in\ BV([0,1]) \end{cases}$$

Let us consider either L^1 or L^∞ norm on $BV([0,1])$. Then

$$F(u) = \Gamma^- \ (L^1) - \lim(F_k) = \Gamma^-(L^\infty) - \lim(F_k)$$

where

$$F(u) = \begin{cases} \int_0^1 \left|\frac{dDu}{d\mu_c}\right|^p d\mu_c & if\ Du \ll \mu_c\ and\ u(0) = 0, u(1) = 1 \\ +\infty & otherwise\ in\ BV([0,1]) \end{cases}$$

Case of a convex integrand with $p - growth$

- Let us again modify the functionals E_k as follows

$$E_k(u) = \int_0^1 f(Du)d\mu_{k,\alpha}$$

where $0 < \lambda_1 < \frac{f(z)}{|z|^p} < \lambda_2$

- As in the previous case $E_k(u)$ is defined on

$$\mathrm{V} = \{u \in W^{1,p}(I) | u(0) = 0, u(1) = 1\}$$

Case of a convex integrand with $p - growth$

The growth conditions on f yield

$$\lambda_1 \left(\rho_p\right)^k \leq \min_{v \in V} E_k\,(v) \leq \lambda_2 \left(\rho_p\right)^k$$

ρ_p is the same constant as before and the minimum is attained on a piecewise affine function, affine on every dyadic interval "of level k«

We then proceed to rescale the functionals via $F_k(u) = \left(\rho_p\right)^{-k} E_k(u)$

However, in this case in order to get the result we need one more hypothesis: we assume there exists $f^{\infty}(z) = \lim_{t \to +\infty} f(tz)/|z|^p$

Then we have

Theorem 2

Let $\{F_k\}_k$ the sequence of functionals previously defined and assume there exists

$$f^{\infty}(z) = \lim_{t\to+\infty} f(tz)/|z|^p$$

Let us consider either L^1 or L^{∞} norm on $BV([0,1])$. Then

$$F(u) = \Gamma^{-}\ (L^1) - \lim(F_k) = \Gamma^{-}(L^{\infty}) - \lim(F_k)$$

where

$$F(u) = \begin{cases} \int_0^1 f^{\infty}\left(\frac{dDu}{d\mu_c}\right) d\mu_c & if\ Du \ll \mu_c\ and\ u(0) = 0, u(1) = 1 \\ +\infty & otherwise\ in\ BV([0,1]) \end{cases}$$

and μ_c is the same of Theorem1'

Counterexample

If in the previous Theorem we choose the following integrand

$$f(z) = z^2\left(\lambda_1 + \sin\left(\ln\left(1 + z^2\right)\right)\right)$$

then the function f is convex (if λ_1 is large enough); however the assumption on f^∞ is not satisfied and the sequence of the values of minima of F_k does not converge.

Result can be extended to product measures

We can obtain some (very simple) extension of previous result in multidimensional case for product measures. As an example, holding true the same notation as before for constants and measures if we set

$$F_k(u) = \begin{cases} (\rho\sigma)^{-k} \int_Q |Du(x)|^2 d\mu_{k,\alpha}(x) \otimes d\mu_{k,\beta}(y) & u \in V \\ +\infty & otherwise\ in\ BV(Q) \end{cases}$$

where $\rho = 4\alpha(1-\alpha)$, $\sigma = 4\beta(1-\beta)$, $Q = [0,1]^2$ and

and $V = \{u \text{ piecewise affine} | u(x,y) = x + y \text{ on } \partial Q\}$ then we can identify the

Γ-limit of F_k as the following functional

$$F(u) = \begin{cases} \int_Q \left|\frac{d(Du_x)}{d\mu_{1-\alpha}}\right|^2 + \left|\frac{d(Du_y)}{d\mu_{1-\beta}}\right|^2 d\mu_{1-\alpha}(x) \otimes d\mu_{1-\beta}(y) & if\ Du_x(\cdot\,, y) \ll \mu_{1-\alpha}, \dots \\ +\infty & otherwise\ in\ BV(Q) \end{cases}$$

What about extension to a truly bidimensional fractal? Some observations