The L^p dissipativity of differential operators

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$$\Omega \subset \mathbb{R}^n$$

$$A : \mathscr{D}(A) \subset L^p(\Omega) \to L^p(\Omega), \quad 1 < p < \infty$$

$$\mathbb{R}\mathrm{e} \int_\Omega \langle Au, u \rangle |u|^{p-2} dx \leqslant 0 \qquad \forall u \in \mathscr{D}(A)$$

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \Delta u, u \rangle dx \leqslant 0 \qquad \forall u \in \mathscr{D}(A)$$

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \Delta u, u \rangle |u|^{p-2} dx \leqslant 0 \qquad \forall u \in \mathscr{D}(A)$$

$$\begin{cases} u' = Au \\ u(0) = u_0 \end{cases} \qquad A : \mathscr{D}(A) \subset X \to X$$

$$T(t) : X \to X \qquad (0 \leqslant t < +\infty)$$

$$T(0) = I, \qquad T(t+s) = T(t)T(s)$$

$$\lim_{t \to 0^+} T(t)x = x, \qquad \forall\, x \in X$$

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}$$

$$\begin{cases} u' = Au \\ u(0) = u_0 \end{cases}$$

$$u(t) = T(t)u_0$$

$$\|T(t)\| \leqslant M\, e^{\omega t} \qquad (M \geqslant 1,\, \omega \geqslant 0)$$

$$\|T(t)\| \leqslant 1 \quad \Longrightarrow \quad \|u(t)\| \leqslant \|u_0\|$$

$$\|u(t)\| \searrow$$

$$\frac{d}{dt}\,\|u(t)\|^p \leqslant 0$$

$$\frac{d}{dt}\int_\Omega |u|^p dx = \frac{d}{dt}\int_\Omega \langle u,u\rangle\,|u|^{p-2}dx = p\,\mathbb{R}\mathrm{e}\int_\Omega \langle u',u\rangle |u|^{p-2}dx$$

$$\mathbb{R}\mathrm{e}\int_\Omega \langle Au,u\rangle |u|^{p-2}dx \leqslant 0 \qquad \forall u \in \mathscr{D}(A)$$

$$\int_\Omega \equiv \int_{\Omega\setminus\{x\in\Omega|u(x)=0\}}$$

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MAZ'YA, V., SOBOLEVSKII, P., On the generating operators of semigroups (Russian), *Uspekhi Mat. Nauk*, 17, 1962, 151–154.
(second order elliptic operators with real coefficients)

KRESIN, G., MAZ'YA, V., Criteria for validity of the maximum modulus principle for solutions of linear parabolic systems, *Ark. Mat.*, 32, 1994, 121–155.
(second order strongly elliptic systems with smooth coefficients; L^p-dissipativity for all p's)

AUSCHER, P., BARTHÉLEMY, L., BÉNILAN, P., OUHABAZ, EL M., Absence de la L^∞-contractivité pour les semi-groupes associés auz opérateurs elliptiques complexes sous forme divergence, *Poten. Anal.*, 12, 2000, 169–189.
(scalar second order operators, L^∞ complex coefficients)

$$\Omega \subset \mathbb{R}^n, a_{kj}, b_k, c_k, a_0 \in L^\infty(\Omega)$$

$$\mathbb{R}\mathrm{e}(a_{kj}(x)\zeta_j\overline{\zeta}_k) \geqslant \eta\,|\zeta|^2, \quad \forall\, \zeta \in \mathbb{C}^n,\ \text{q.o. } x \in \Omega\ (\eta > 0)$$

$$A_V u(x) =$$
$$\partial_j(a_{kj}(x)\partial_k u(x)) - b_k(x)\partial_k u(x) + \partial_k(c_k(x)u(x)) - a_0(x)u(x)$$

Theorem (AUSCHER, BARTHÉLEMY, BÉNILAN, OUHABAZ)

The semigorup $\left(e^{-tA_{H_0^1}}\right)_{t\geqslant 0}$ *is* L^∞*-contractive if, and only if,:*
(i) $\mathbb{I}m(a_{kj} + a_{jk}) = 0,\ j, k = 1, \ldots, n;$
(ii) $f_0 = \mathbb{R}e\ a_0 - \partial_j(\mathbb{R}e\ c_j)$ *is a positive Radon measure on* Ω*;*
(iii) $f_k = \partial_j(\mathbb{I}m\ a_{kj}) \in L^1_{loc}(\Omega),\ k = 1, \ldots, n;$
(iv) $\mathbb{R}e(a_{kj})\xi_k\xi_j + (\mathbb{I}m(c_j - b_j) + f_j)\xi_j + f_{0,r} \geqslant 0, \quad q.o.\ x \in \Omega, \forall\, \xi \in \mathbb{R}^n.$

MAZ'YA, V., SOBOLEVSKII, P., On the generating operators of semigroups (Russian), *Uspekhi Mat. Nauk*, 17, 1962, 151–154.
(second order elliptic operators, real coefficients)

KRESIN, G., MAZ'YA, V., Criteria for validity of the maximum modulus principle for solutions of linear parabolic systems, *Ark. Mat.*, 32, 1994, 121–155.
(sistemi fortemente ellittici del secondo ordine a coefficienti regolari, dissipatività in L^p simultaneamente per tutti i p)

AUSCHER, P., BARTHÉLEMY, L., BÉNILAN, P., OUHABAZ, EL M., Absence de la L^∞-contractivité pour les semi-groupes associés auz opérateurs elliptiques complexes sous forme divergence, *Poten. Anal.*, 12, 2000, 169–189.
(operatori scalari del secondo ordine, coefficienti complessi L^∞)

LANGER, M., MAZ'YA, V. On L^p-contractivity of semigroups generated by linear partial differential operators, *J. of Funct. Anal.*, 164, 1999, 73–109.

Theorem (Langer, Maz'ya)

If $\Omega \subset \mathbb{R}^n$ is open and $1 \leqslant p < \infty$, $p \neq 2$, no linear partial differential operator of order higher than two which contains $(C_0^\infty(\Omega))^N$ in its domain of definition can generate a contraction semigroup on $(L^p(\Omega))^N$.

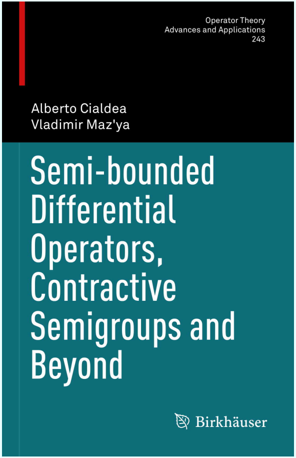
Theorem (Langer, Maz'ya)

Let $1 < p < \infty$, $p \neq 2$ and suppose that $C_0^\infty(\Omega)$ is a subset of the domain $D(A)$ of the linear partial differential operator A. Assume furthermore that A has $L^1_{\rm loc}$-coefficients and that the related Cauchy problem is well-posed for all nonnegative initial data in $D(A)$. If

$$\frac{d}{dt}\|s(t)\|_p \Big|_{t=0^+} \leqslant 0$$

for every $s(0) \in (D(A))^+$, then either A is of order 0, 1 or 2, or A is of order 4 and $\frac{3}{2} \leqslant p \leqslant 3$.

Amann, Brezis, Daners, Davies, Karrmann, Kovalenko, Langer, Liskevich, Ouhabaz, Robinson, Sinestrari, Sobol, Strichartz, Vogt, ...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$$\Omega \subset \mathbb{R}^n, \quad C_0(\Omega) = \{u \in C(\Omega) | \operatorname{spt} u \subset\subset \Omega\}, \quad C_0^1(\Omega) = C^1(\Omega) \cap C_0(\Omega)$$

$$\mathscr{A} = \{a^{hk}\},\ a^{hk} \in (C_0(\Omega))^*,\ \mathscr{A}^t = \text{transposed matrix of } \mathscr{A},\ \mathscr{A}^* = \overline{\mathscr{A}}^t$$

$$\mathbf{b} = (b_1, \dots, b_n), \quad \mathbf{c} = (c_1, \dots, c_n), \quad b_j, c_j \in (C_0(\Omega))^*, \quad a \in (C_0^1(\Omega))^*$$

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A} \nabla u, \nabla v \rangle - \langle \mathbf{b} \nabla u, v \rangle + \langle u, \overline{\mathbf{c}} \nabla v \rangle - a \langle u, v \rangle)$$

$$(u,v) \in C_0^1(\Omega) \times C_0^1(\Omega)$$

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A}\,\nabla u, \nabla v\rangle - \langle \mathbf{b}\nabla u, v\rangle + \langle u, \overline{\mathbf{c}}\nabla v\rangle - a\langle u, v\rangle)$$

$$\mathscr{L} : C_0^1(\Omega) \times C_0^1(\Omega) \to \mathbb{C}$$

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + \nabla^t(\mathbf{c}u) + au$$

$$A : C_0^1(\Omega) \to (C_0^1(\Omega))^*$$

$$\mathscr{L}(u,v) = -\int_\Omega \langle Au, v\rangle\,, \qquad \forall\ u,v \in C_0^1(\Omega)$$

Definition

Let $1 < p < \infty$. The form $\mathscr{L}$ is called *L^p-dissipative* if for all $u \in C_0^1(\Omega)$

$$\mathbb{R}\mathrm{e}\ \mathscr{L}(u, |u|^{p-2}u) \geqslant 0 \qquad \text{if } p \geqslant 2$$
$$\mathbb{R}\mathrm{e}\ \mathscr{L}(|u|^{p'-2}u, u) \geqslant 0 \qquad \text{if } 1 < p < 2$$

$$(p' = \tfrac{p}{p-1})$$

$$\mathscr{L}(u, |u|^{p-2}u) = -\int_\Omega \langle Au, u\rangle |u|^{p-2}$$

$(|u|^{p-2}u \in C_0^1(\Omega)$ for $p \geqslant 2$ and $u \in C_0^1(\Omega))$

$$v = |u|^{p-2}u \implies u = |v|^{p'-2}v$$

Lemma

The form $\mathscr{L}$ is L^p-dissipative if, and only if,

$$\mathbb{R}e \int_\Omega \Big[\langle \mathscr{A} \nabla v, \nabla v \rangle - (1-2/p) \langle (\mathscr{A} - \mathscr{A}^*) \nabla(|v|), |v|^{-1} \overline{v} \nabla v \rangle -$$

$$(1-2/p)^2 \langle \mathscr{A} \nabla(|v|), \nabla(|v|) \rangle \Big] + \int_\Omega \langle \mathbb{I}m(\mathbf{b}+\mathbf{c}), \mathbb{I}m(\overline{v} \nabla v) \rangle +$$

$$\int_\Omega \mathbb{R}e \, (\nabla^t(\mathbf{b}/p - \mathbf{c}/p') - a)|v|^2 \geqslant 0,$$

for any $v \in C_0^1(\Omega)$.

Corollary

If the form $\mathscr{L}$ is L^p-dissipative, then

$$\langle \mathbb{R}e\, \mathscr{A}\, \xi, \xi \rangle \geqslant 0$$

for any $\xi \in \mathbb{R}^n$.

Corollary

If

$$\frac{4}{p\,p'}\langle \mathbb{R}e\, \mathscr{A}\, \xi, \xi \rangle + \langle \mathbb{R}e\, \mathscr{A}\, \eta, \eta \rangle + 2\langle (p^{-1}\mathbb{I}m\, \mathscr{A} + p'^{-1}\mathbb{I}m\, \mathscr{A}^*)\xi, \eta\rangle + \langle \mathbb{I}m(\mathbf{b}+\mathbf{c}), \eta\rangle + \mathbb{R}e\left[\nabla^t\left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] \geqslant 0$$

for any $\xi, \eta \in \mathbb{R}^n$, then the form $\mathscr{L}$ is L^p-dissipative.

$$\mathscr{A} = \begin{pmatrix} 1 & i\gamma \\ -i\gamma & 1 \end{pmatrix}$$

$$\gamma \in \mathbb{R}$$

$$(\eta_1 + \gamma\xi_2)^2 + (\eta_2 - \gamma\xi_1)^2 - (\gamma^2 - 4/(pp'))|\xi|^2 \not\geq 0$$

$$A = \Delta$$

N.B. $\mathbb{I}m\, \mathscr{A}^t \neq \mathbb{I}m\, \mathscr{A}$

$$Au = \nabla^t(\mathscr{A}\,\nabla u) \qquad (a^{hk} \in (C_0(\Omega))^*)$$

Theorem

Let the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ *be symmetric, i.e.* $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. *The form*

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla v\rangle$$

is L^p*-dissipative if and only if*

$$|p-2|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle$$

for any $\xi \in \mathbb{R}^n$, *where* $|\cdot|$ *denotes the total variation.*

$$|p-2|\,|\langle \mathbb{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle\,, \qquad \forall\,\xi\in\mathbb{R}^n$$

$\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle \geqslant 0$ for all $\xi\in\mathbb{R}^n \Longrightarrow$

- A is L^2-dissipative
- if A is a real coefficient operator, A is L^p-dissipative for any p

$$|p-2|\,|\langle \mathbb{I}m\, \mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}e\, \mathscr{A}\,\xi,\xi\rangle\,, \qquad \forall\, \xi\in\mathbb{R}^n$$

$$\Updownarrow$$

$$\frac{4}{p\,p'}\langle \mathbb{R}\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathbb{R}\,\mathscr{A}\,\eta,\eta\rangle + 2\langle (p^{-1}\mathbb{I}m\,\mathscr{A} + p'^{-1}\mathbb{I}m\,\mathscr{A}^*)\xi,\eta\rangle \geqslant 0$$

for any $\xi,\eta\in\mathbb{R}^n$.

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DINDOŠ, M., PIPHER, J.: Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem, *Adv. Math.*, 341 (2019), 255–298.

DAVID, G., FENEUIL, J., MAYBORODA, S.: A new elliptic measure on lower dimensional sets, arXiv:1807.07035v1.

M. EGERT,: On p-elliptic divergence form operators and holomorphic semigroups, arXiv:1812.09154v1.

$$\Omega \subset\subset \mathbb{R}^2, \quad \sigma : \Omega \to \mathbb{R}, \ \sigma \in C_0^2(\Omega), \quad \sigma \not\equiv 0, \quad \lambda \in \mathbb{R}$$

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix}$$

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u)$$

$$|p-2|\,|\langle \mathbb{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle\,, \qquad \forall\, \xi \in \mathbb{R}^n$$

$$0 \leqslant 2\sqrt{p-1}\,|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^n$$

$\exists\ \lambda, \sigma$ such that A is not L^2-dissipative

$$\mathbb{R}\mathrm{e}\int_\Omega ((\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u)\partial_1\overline{u} + (-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u)\partial_2\overline{u})\,dx \geqslant 0$$

$$\forall\ u \in C_0^1(\Omega)$$

$$\int_\Omega |\nabla u|^2 dx - 2\lambda\int_\Omega \partial_1(\sigma^2)\mathbb{I}m(\partial_1\overline{u}\,\partial_2 u)\,dx \geqslant 0$$

$$\forall\ u \in C_0^1(\Omega)$$

$$u = \sigma\ \exp(itx_2) \quad (t\in\mathbb{R})$$

$$t^2\int_\Omega \sigma^2 dx - t\lambda\int_\Omega (\partial_1(\sigma^2))^2 dx + \int_\Omega |\nabla\sigma|^2 dx \geqslant 0, \quad \forall\ t\in\mathbb{R}$$

$$\left(\sigma\in C_0^2(\Omega),\ \sigma\not\equiv 0 \Rightarrow \int_\Omega (\partial_1(\sigma^2))^2 dx \neq 0\right)$$

$$\Omega \subset\subset \mathbb{R}^2, \quad \sigma \in C_0^2(\Omega), \quad \sigma \not\equiv 0, \quad \lambda \in \mathbb{R}$$

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix}$$

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u)$$

$$|p-2|\,|\langle \mathbb{I}m\,\mathscr{A}\,\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi, \xi\rangle\,, \qquad \forall\, \xi \in \mathbb{R}^n$$

$$0 \leqslant 2\sqrt{p-1}\,|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^n$$

A is not L^2-dissipative

$$Au = \Delta u - i\lambda(\partial_{21}(\sigma^2)\,\partial_1 u - \partial_{11}(\sigma^2)\,\partial_2 u)$$

$$Au = \Delta u + a(x)u, \qquad \Omega \subset\subset \mathbb{R}^n$$

$$\mathbb{R}\mathrm{e}\, a \leqslant \lambda_1 \quad \Longrightarrow \quad L^2 - \text{dissipativity}$$

$$A = \Delta + \mu \qquad (\mu \geqslant 0)$$

A is L^p-dissipative if, and only if,

$$\int_\Omega |w|^2 d\mu \leqslant \frac{4}{pp'} \int_\Omega |\nabla w|^2 dx, \qquad \forall\ w \in C_0^\infty(\Omega)$$

$$\text{Nec. Cond.} \qquad \frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leqslant \frac{4}{pp'}\,, \qquad \forall\ F \subset \Omega$$

$$\mathrm{cap}_\Omega(F) = \inf \left\{ \int_\Omega |\nabla u|^2 dx \ :\ u \in C_0^\infty(\Omega),\ u \geqslant 1 \text{ on } F \right\}$$

$$\mu(F) \leqslant \int_F |u|^2 d\mu \leqslant \int_\Omega |u|^2 d\mu \leqslant \frac{4}{pp'} \int_\Omega |\nabla u|^2 dx$$

$$\text{Suff. Cond.} \qquad \frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leqslant \frac{1}{pp'}\,, \qquad \forall\ F \subset \Omega,$$

(Maz'ya)

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + au, \quad a^{hk}, b_j, a \in \mathbb{C}, \qquad \mathscr{A}^t = \mathscr{A}$$

Theorem

Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. The operator A is L^p-dissipative if and only if there exists a real constant vector V such that

$$2\mathbb{R}e\,\mathscr{A}\,V + \mathbb{I}m\mathbf{b} = 0$$

$$\mathbb{R}ea + \langle \mathbb{R}e\,\mathscr{A}\,V, V\rangle \leqslant 0$$

and the inequality

$$|p-2|\,|\langle \mathbb{I}m\,\mathscr{A}\,\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}e\,\mathscr{A}\,\xi, \xi\rangle$$

holds for any $\xi \in \mathbb{R}^n$.

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + au, \quad a^{hk}, b_j, a \in \mathbb{C}, \qquad \mathscr{A}^t = \mathscr{A}$$

Corollary

Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. Let us suppose that the matrix $\mathbb{R}e\,\mathscr{A}$ is not singular. The operator A is L^p-dissipative if and only if

$$|p-2|\,|\langle \mathbb{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}e\,\mathscr{A}\,\xi,\xi\rangle \qquad \forall\, \xi \in \mathbb{R}^n$$

holds and

$$4\mathbb{R}e\,a \leqslant -\langle (\mathbb{R}e\,\mathscr{A})^{-1}\mathbb{I}m\mathbf{b}, \mathbb{I}m\mathbf{b}\rangle\ .$$

(Kresin, Maz'ya, 1994)

The angle of L^p-dissipativity

$$A = \nabla^t(\mathscr{A}(x)\nabla), \qquad a_{hk} \in L^1_{\mathrm{loc}}(\Omega), \qquad \mathscr{A}^t = \mathscr{A}$$

Theorem

Let us suppose that the form $\mathscr{L}$ is L^p-dissipative. The form $z\mathscr{L}$ is L^p-dissipative if, and only if,

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+$$

$$\Lambda_1 = \operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \frac{\langle \mathbb{I}m\, \mathscr{A}(x)\xi,\xi\rangle}{\langle \mathbb{R}e\, \mathscr{A}(x)\xi,\xi\rangle}, \quad \Lambda_2 = \operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \frac{\langle \mathbb{I}m\, \mathscr{A}(x)\xi,\xi\rangle}{\langle \mathbb{R}e\, \mathscr{A}(x)\xi,\xi\rangle}$$

where

$$\Xi = \{(x,\xi) \in \Omega \times \mathbb{R}^n \mid \langle \mathbb{R}e\, \mathscr{A}(x)\xi,\xi\rangle > 0\}.$$

$$\vartheta_- = \begin{cases} \operatorname{arccot}\left(\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}+|p-2|\Lambda_1}\right) - \pi & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_1) - \pi & \text{if } p = 2 \end{cases}$$

$$\vartheta_+ = \begin{cases} \operatorname{arccot}\left(-\frac{2\sqrt{p-1}}{|p-2|} + \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}-|p-2|\Lambda_2}\right) & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_2) & \text{if } p = 2. \end{cases}$$

$(0 < \operatorname{arccot} y < \pi,\ \operatorname{arccot}(+\infty) = 0,\ \operatorname{arccot}(-\infty) = \pi)$

$$\mathbb{I}m\,\mathscr{A} \equiv 0 \quad \Longrightarrow \quad \Lambda_1 = \Lambda_2 = 0$$

$$\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{2\sqrt{p-1}|p-2|} = -\frac{|p-2|}{2\sqrt{p-1}}$$

The form $z\mathscr{L}$ is L^p-dissipative if, and only if,

$$\operatorname{arccot}\left(-\frac{|p-2|}{2\sqrt{p-1}}\right) - \pi \leqslant \arg z \leqslant \operatorname{arccot}\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

i.e.

$$|\arg z| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right)$$

Fattorini, 1983; Okazawa, 1991.

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla\nabla^t u$$

(either $\nu > 1$ or $\nu < 1/2$)

$$\mathscr{L}(u,v) = -\int_\Omega (\langle \nabla u, \nabla v\rangle + (1-2\nu)^{-1}\nabla^t u\, \nabla^t v)\, dx$$

$\mathscr{L}$ is L^p-dissipative in $\Omega \subset \mathbb{R}^n$ if for all $u \in (C_0^1(\Omega))^n$ ($p' = p/(p-1)$)

$$-\int_\Omega (\langle \nabla u, \nabla(|u|^{p-2}u)\rangle + (1-2\nu)^{-1}\nabla^t u\, \nabla^t(|u|^{p-2}u))\, dx \leqslant 0, \quad \text{if } p \geqslant 2,$$
$$-\int_\Omega (\langle \nabla u, \nabla(|u|^{p'-2}u)\rangle + (1-2\nu)^{-1}\nabla^t u\, \nabla^t(|u|^{p'-2}u))\, dx \leqslant 0, \quad \text{if } p < 2.$$

Lemma

Let Ω be a domain of $\mathbb{R}^n$. The form $\mathscr{L}$ is L^p-dissipative if and only if

$$\int_\Omega [C_p|\nabla|v||^2 - \sum_{j=1}^n |\nabla v_j|^2 + \gamma\, C_p\, |v|^{-2}|v_h\partial_h|v||^2 - \gamma\, |\nabla^t v|^2]\, dx \leqslant 0$$

for any $v \in (C_0^1(\Omega))^n$, where

$$C_p = (1-2/p)^2, \qquad \gamma = (1-2\nu)^{-1}.$$

n=2

Lemma

Let Ω be a domain of $\mathbb{R}^2$. If the form $\mathscr{L}$ is L^p-dissipative, we have

$$C_p[|\xi|^2 + \gamma \langle \xi, \omega \rangle^2] \langle \lambda, \omega \rangle^2 - |\xi|^2 |\lambda|^2 - \gamma \langle \xi, \lambda \rangle^2 \leqslant 0$$

for any ξ, λ, $\omega \in \mathbb{R}^2$, $|\omega| = 1$.

n=2

Theorem

The form $\mathscr{L}$ is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,.$$

Corollary

Let E be the two-dimensional elasticity operator with domain $(W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega))^2$, $\Omega\subset\subset\mathbb{R}^2$, $\partial\Omega\in C^2$. The operator E is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,.$$

$$\boxed{n > 2}$$

Necessary Condition

Theorem

Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^2$. Suppose $\nu = \nu(x)$ is a continuos function defined in Ω, such that

$$\inf_{x\in\Omega} |2\nu(x) - 1| > 0.$$

If the elasticity operator is L^p-dissipative in Ω, then

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \inf_{x\in\Omega} \frac{2(\nu(x) - 1)(2\nu(x) - 1)}{(3 - 4\nu(x))^2} .$$

$$\boxed{n > 2}$$

Sufficient Condition

Theorem

Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^2$. *If*

$$(1-2/p)^2 \leqslant \begin{cases} \dfrac{1-2\nu}{2(1-\nu)} & \text{if } \nu < 1/2 \\[2ex] \dfrac{2(1-\nu)}{1-2\nu} & \text{if } \nu > 1. \end{cases}$$

the operator E *is* L^p*-dissipative.*

Open problem.

If $n > 2$, is the necessary condition

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \frac{2(\nu - 1)(2\nu - 1)}{(3 - 4\nu)^2}$$

also sufficient ?

MAZ'YA, V. : Seventy five (thousand) unsolved problems in Analysis and partial differential equations, *Integr. Equ. Oper. Theory* (2018)

A is accretive if $-A$ is dissipative
(KATO) if A is accretive in a Hilbert space, then A^α $(0 < \alpha < 1)$ are accretive.

Theorem

Let $0 < \alpha < 1$, $1 < p < \infty$. We have, for any $u \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \langle (-\Delta)^\alpha u, u \rangle |u|^{p-2} dx \geqslant \frac{2\, c_\alpha}{p\, p'} \, \| |u|^{p/2} \|^2_{\mathcal{L}^{\alpha,2}(\mathbb{R}^n)} \, ,$$

where

$$c_\alpha = -\pi^{-n/2} 4^\alpha \Gamma(\alpha + n/2)/\Gamma(-\alpha) > 0.$$

$$\|v\|_{\mathcal{L}^{\alpha,2}} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |v(x+t) - v(x)|^2 \frac{dxdt}{|t|^{n+2\alpha}} \right)^{1/2}$$

$$\mathscr{B}^h = \{b_{ij}^h\},\ \mathscr{C}^h = \{c_{ij}^h\},\ h = 1, \dots, n;\ 1 \leqslant i, j \leqslant m$$

$$b_{ij}^h, c_{ij}^h \in (C_0(\Omega))^*$$

$$\mathscr{D} = \{d_{ij}\},\ 1 \leqslant i, j \leqslant m;\ d_{ij} \in (C_0^1(\Omega))^*$$

$$\mathscr{L}(u, v) = \int_\Omega \langle \mathscr{B}^h\, \partial_h u, v \rangle - \langle \mathscr{C}^h\, u, \partial_h v \rangle + \langle \mathscr{D}\, u, v \rangle$$

$$\mathscr{L} : (C_0^1(\Omega))^m \times (C_0^1(\Omega))^m \to \mathbb{C}$$

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{B}^h\, \partial_h u, v\rangle - \langle \mathscr{C}^h\, u, \partial_h v\rangle + \langle \mathscr{D}\, u, v\rangle$$

$$Eu = \mathscr{B}^h\, \partial_h u + \partial_h(\mathscr{C}^h\, u) + \mathscr{D}\, u$$

Theorem

The form $\mathscr{L}$ is L^p-dissipative in Ω if, and only if,

$$\int_\Omega \Big((1-2/p)|v|^{-2}\mathbb{R}e\langle \mathscr{B}^h\, v, v\rangle \mathbb{R}e\langle v, \partial_h v\rangle \; - \mathbb{R}e\langle \mathscr{B}^h\, \partial_h v, v\rangle +$$
$$(1-2/p)|v|^{-2}\mathbb{R}e\langle \mathscr{C}^h\, v, v\rangle \mathbb{R}e\langle v, \partial_h v\rangle \; + \mathbb{R}e\langle \mathscr{C}^h\, v, \partial_h v\rangle - \mathbb{R}e\langle \mathscr{D}\, v, v\rangle \Big) \geqslant 0$$

for any $v \in (C_0^1(\Omega))^m$.

$$Eu = \mathscr{B}^h \, \partial_h u + \mathscr{D} \, u, \quad \mathscr{B}^h, \, \mathscr{D}, \, \partial_h \, \mathscr{B}^h \in L^1_{loc}(\Omega)$$

Theorem

The form $\mathscr{L}$ is $L^p -$ dissipative if, and only if, the following conditions are satisfied:

$$\mathscr{B}^h(x) = (\mathscr{B}^h)^*(x), \qquad \textit{if } p = 2,$$
$$\mathscr{B}^h(x) = b_h(x) \, I, \qquad \textit{if } p \neq 2,$$

for almost any $x \in \Omega$ and $h = 1, \ldots, n$. Here b_h are real locally integrable functions ($1 \leqslant h \leqslant n$). Moreover

$$\mathbb{R}e \langle (p^{-1} \partial_h \, \mathscr{B}^h(x) - \mathscr{D}(x)) \zeta, \zeta \rangle \geqslant 0$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

Thank you for your attention !